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On the Connection of an Abstract Set, with Applications to the Theory of Functions of a General Variable.

By Arthur D. Pitcher.

In that portion of the thesis of Fréchet[*] which relates to the theory of functions[†] of a general variable there appear theorems which secure for a continuous function of such a general variable the more fundamental properties possessed by continuous functions of an ordinary real variable. For instance, given a system $(\mathfrak{Q}; L)$, *i. e.*, a set $\mathfrak{Q}$ of elements q on which a limit L satisfying certain postulates is defined, a sufficient condition that every continuous function on $\mathfrak{Q}$ (1) be bounded, (2) attain its bounds, is that $\mathfrak{Q}$ be extremal[‡] (compact and closed). A sufficient condition that every continuous function on $\mathfrak{Q}$ (3) assume every value between each pair of its values is that $\mathfrak{Q}$ be "continuous," where the term "continuous" is defined in terms of a certain postulated correspondence between subsets of $\mathfrak{Q}$ and the linear interval (0 1).[§] Later, for a system $(\mathfrak{Q}; \delta)$ (*i. e.*, a set $\mathfrak{Q}$ of elements on which a distance function δ satisfying certain postulates is defined) it is shown that a necessary and sufficient condition for (1) and (2) is that $\mathfrak{Q}$ be extremal.[||] Fréchet does not consider (3) further. It will be recognized that (3) is a property not less important than (2) and (1).

In his dissertation entitled "A Contribution to the Foundations of Fréchet's Calcul Fonctionnel,"[¶] T. H. Hildebrandt gives an excellent analysis of the Fréchet theory, securing many of the Fréchet theorems under hypotheses milder than those used by Fréchet. Hilderbrandt considers especially systems $(\mathfrak{Q}; K)$, replacing the δ of Fréchet by a relation K between the pairs of elements of $\mathfrak{Q}$ and the single elements of the class $[m]$ of positive and negative integers m.[**]

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[*] "Sur Quelques Points du Calcul Fonctionnel," Paris, reprinted in *Rendiconti del Circolo Matematico di Palermo,* Vol. XXII, pp. 1–64.

[†] The term "function" is used here and in the sequel in the sense of real-valued, single-valued function.

[‡] Fréchet, *loc. cit.*, § 11.

[§] Fréchet, *loc. cit.*, § 12.

[||] Fréchet, *loc. cit.*, § 51.

[¶] American Journal of Mathematics, Vol. XXXIV, pp. 237–290. We refer to this paper in the sequel as Hildebrandt.

[**] It is very convenient and precise to speak of this relation as a relation on $\mathfrak{Q}\mathfrak{Q}\mathfrak{I}$, where $\mathfrak{I}$ is the class of positive and negative integers. Cf. § 69 of the memoir by E. H. Moore entitled "Introduction to a Form of General Analysis," *The New Haven Mathematical Colloquium,* New Haven, 1910. The relation K_2 introduced by Moore is essentially the K relation used by Hildebrandt.

He conditions the K by postulates sufficient to secure the theorems of Fréchet and points out the precise relation between the K and the δ.[*] In particular Hildebrandt indicates that for a system $(\mathfrak{Q}; K)$, where K is properly conditioned, a necessary and sufficient condition for (1) and (2) above is that $\mathfrak{Q}$ be extremal. He does not consider (3).

It is the purpose of the present paper to give, in terms of the K relation of Hildebrandt, a definition of the notion *connection*[†] as applied to a general range; to give some simple but general theorems in the theory of functions of a general variable; to give for systems $(\mathfrak{Q}; K)$, where K is properly conditioned and $\mathfrak{Q}$ is extremal, a *necessary* and *sufficient* condition that every continuous function on $\mathfrak{Q}$ possess the property (3) above; and finally to give a set of conditions *necessary* and *sufficient* that every continuous function on $\mathfrak{Q}$ possess the properties (1), (2), (3). The theorems are applicable to the Fréchet systems $(\mathfrak{Q}; \delta)$.

Consider a class $\mathfrak{Q} \equiv [q]$ of elements q, unconditioned except by the postulates of the sequel, and the class $[m]$ of all integers, positive and negative. Consider also a relation K between pairs of elements $q_1 q_2$ of $\mathfrak{Q}$ and single integers m. Denote the fact that $q_1 q_2$ and m are in the relation K by the symbol $K q_1 q_2 m$. The relation K may have a variety of properties. (Cf. Hildebrandt, § 4.) For instance, the relation K may be such that the following condition is satisfied:

$$\text{If } m_0 \leq m_1, \text{ then } K q_1 q_2 m_1 \text{ implies } K q_1 q_2 m_0. \tag{1}$$

In the sequel a K relation for which (1) holds is denoted by K^1, while K denotes the general K relation.

In a system $(\mathfrak{Q}; K)$ the sequence $\{q_n\}$ of elements of $\mathfrak{Q}$ is said to have the element q as a limit if, for every m, there is an n_m such that $n \geq n_m$ implies $K q_n q m$.[‡]

A function μ is said to be continuous on the range $\mathfrak{Q} \equiv [q]$ of a system $(\mathfrak{Q}; K)$ if, for every q and positive number e, there is an m_{qe} such that $K q_1 q m_{qe}$ implies $|\mu(q_1) - \mu(q)| \leq e$.

The function μ is said to be uniformly continuous if, for every e, there is an m_e such that $K q_1 q_2 m_e$ implies $|\mu(q_1) - \mu(q_2)| \leq e$.

Two elements $\hat{q}$ and $\tilde{q}$ of the range $\mathfrak{Q}$ of a system $(\mathfrak{Q}; K)$ are *connected for a given m* if there is a finite set $q_1, q_2, \ldots, q_k$ such that $\hat{q} = q_1$ and $\tilde{q} = q_k$, and such that, for each pair $q_i q_{i+1}$ of elements of the set, we have either $Kq_i q_{i+1} m$ or else $Kq_{i+1} q_i m$.

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[*] Hildebrandt, § 6.

[†] The idea of the connection of an abstract range was first brought to my attention by Professor E. H. Moore at the time when I had the privilege of attending his lectures on "General Analysis" at the University of Chicago.

F. Riesz, in his article "Stetigkeitsbegriff und Abstracte Mengenlehre," *Atti del IV. Congresso Internazionale dei Matematici*, Rome, 1908, discusses the connection of subsets of an abstract point set.

[‡] Hildebrandt, § 7.

The elements qq are *connected* if they are connected for every m.

The range $\mathfrak{Q}$ is said to be *connected* when each pair $q_1 q_2$ of elements of $\mathfrak{Q}$ is connected.

The set $\mathfrak{Q}$ of a system $(\mathfrak{Q}; K)$ is *extremal** if every infinitude of elements of $\mathfrak{Q}$ gives rise to at least one limit element of $\mathfrak{Q}$. That is, if $\mathfrak{Q}_1$ is a subclass of $\mathfrak{Q}$ containing an infinite number of elements, then there is a sequence $\{q_{1n}\}$ of $\mathfrak{Q}_1$ and an element q of $\mathfrak{Q}$ such that $\underset{n}{L} q_n = q$.

The following propositions relative to systems $(\mathfrak{Q}; K)$ are easily proved.

If $q_1 q_2$ are connected for m_0 and $q_2 q_3$ are connected for m_0, then $q_1 q_3$ are connected for m_0.

If $q_1 q_2$ are connected and $q_2 q_3$ are connected, then $q_1 q_3$ are connected.

If $\mathfrak{R}$ is a connected subclass of $\mathfrak{Q}$, then $\mathfrak{R} + \mathfrak{R}'$ is connected. ($\mathfrak{R}'$ is the derived set of $\mathfrak{R}$.)

If $\mathfrak{Q}_1$ is the class of all elements connected with a given q_1, then $\mathfrak{Q}_1$ is closed. (A subset $\mathfrak{Q}_1$ of $\mathfrak{Q}$ is closed in case all the elements of $\mathfrak{Q}$ which are limits of sequences of $\mathfrak{Q}_1$ are also elements of $\mathfrak{Q}_1$.)

If $\mathfrak{Q}_1$ is the class of elements composed of q_1 and all elements q connected with q_1, and $\mathfrak{Q}_2$ the class of elements composed of q_2 and all elements connected with q_2, then the necessary and sufficient condition that $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ have a common element, is that q_1 and q_2 be connected. In fact, if $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ have a single element in common, they are identical.

The following proposition is for systems $(\mathfrak{Q}; K^1)$,

If q_1 and q_2 are connected for m, they are connected for every $m_0 \leq m$.

Denote by $\mathfrak{A}_\mu$ the set of real numbers composed of the functional values of μ, and by $(\mathfrak{A}_\mu)_1$ the set $\mathfrak{A}_\mu$ plus its derived set. The following theorem is relative to a system $(\mathfrak{Q}; K)$.

THEOREM I. (a) *If* q_1 *and* q_2 *are connected and* μ *is a uniformly continuous function on* $\mathfrak{Q}$ *such that* $\mu(q_1) < h < \mu(q_2)$, *then* h *belongs to* $(\mathfrak{A}_\mu)_1$. (b) *If* $\mathfrak{Q}$ *is connected and* k *is any number between two values of a uniformly continuous function* μ *on* $\mathfrak{Q}$, *then* k *belongs to* $(\mathfrak{A}_\mu)_1$.

The second part of the theorem is an immediate corollary of the first part. To prove the first part it is sufficient to prove that if $\mu(q_1) < 0 < \mu(q_2)$, then 0 belongs to $(\mathfrak{A}_\mu)_1$. The method of proof is standard. (Cf. Goursat-Hedrick, "Mathematical Analysis," p. 145.) Denote by $\hat{\mathfrak{Q}} \equiv [\hat{q}]$ the class of all elements $\hat{q}$ such that $\mu(\hat{q}) > 0$, and by $\bar{\mathfrak{Q}} \equiv [\bar{q}]$ the class of all elements $\bar{q}$ such that $\mu(\bar{q}) < 0$. Denote by $\hat{\mathfrak{A}}_\mu$ the set of all positive functional values of μ, and by $\bar{\mathfrak{A}}_\mu$ the set of all negative functional values of μ. Then the lower limit of $\hat{\mathfrak{A}}_\mu$ is either 0 or $\varepsilon > 0$. Suppose the latter to be true. Then $|\mu(\hat{q}) - \mu(\bar{q})| \geq \varepsilon$ for every $\hat{q}\bar{q}$. But since $q_1 q_2$ are connected, there is for every m a pair $\hat{q}_m \bar{q}_m$ such that either $K\hat{q}_m\bar{q}_m m$ or $K\bar{q}_m\hat{q}_m m$. Since μ is uniformly continuous, there is for $\varepsilon/2$ an $m_{\epsilon/2}$ such that $Kq_1q_2m_{\epsilon/2}$ implies $|\mu(q_1) - \mu(q_2)| < \varepsilon/2$. From above there is a pair $\hat{q}_{m_{\epsilon/2}}\bar{q}_{m_{\epsilon/2}}$ such that $K\hat{q}_{m_{\epsilon/2}}\bar{q}_{m_{\epsilon/2}}m_{\epsilon/2}$, and such that $|\mu(\hat{q}_{m_{\epsilon/2}}) - \mu(\bar{q}_{m_{\epsilon/2}})| > \varepsilon/2$. Thus we have a contradiction.

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* Cf. Hildebrandt, § 10.

THEOREM II. (a) *If the class* $\mathfrak{Q}$ *of a system* $(\mathfrak{Q};\, K)$ *is extremal and* q_1, q_2 *are two connected elements of* $\mathfrak{Q}$, *and* μ *is a uniformly continuous function on* $\mathfrak{Q}$ *such that* $\mu(q_1) < h < \mu(q_2)$, *then* h *belongs to* $\mathfrak{A}_\mu$. (b) *If* $\mathfrak{Q}$ *is extremal and connected, then every uniformly continuous function* μ *on* $\mathfrak{Q}$ *assumes every value between each pair of its values.*

The second part of the theorem follows from the first. The first part may be seen readily as follows:* By the previous theorem h belongs to $(\mathfrak{A}_\mu)_1$. Then either h belongs to $\mathfrak{A}_\mu$ or else there is a sequence $\{h_n\}$ of numbers of $\mathfrak{A}_\mu$ such that $\underset{n}{L} h_n = h$. This, combined with the fact that $\mathfrak{Q}$ is extremal, implies that there is a sequence $\{q_n\}$ of $\mathfrak{Q}$ and an element q_0 of $\mathfrak{Q}$ such that $\underset{n}{L} q_n = q_0$ and $\underset{n}{L} \mu(q_n) = h$. It follows readily from this and the continuity of μ that $\mu(q_0) = h$.

If $\mathfrak{Q}$ is extremal, then every continuous function on $\mathfrak{Q}$ is bounded and attains its least upper and greatest lower bounds. (Cf. Hildebrandt, § 19.) From this and the previous theorem we have:

THEOREM III. *If a class* $\mathfrak{Q}$, *of a system* $(\mathfrak{Q};\, K)$, *is extremal and connected, then every uniformly continuous function* μ *on* $\mathfrak{Q}$ *is bounded and assumes its least upper and greatest lower bounds and every value between these bounds.*

THEOREM IV. (a) *A neccessary and sufficient condition that every function* μ *uniformly continuous on* $\mathfrak{Q}$, *of a system* $(\mathfrak{Q};\, K^1)$, *be such that if* $\mu(q_1) < h < \mu(q_2)$ *then* h *belongs to* $(\mathfrak{A}_\mu)_1$, *is that* q_1 *and* q_2 *be connected.* (b) *A necessary and sufficient condition that every function* μ *uniformly continuous on* $\mathfrak{Q}$, *of a system* $(\mathfrak{Q},\, K^1)$, *be such that every value between any two of its values belongs to* $(\mathfrak{A}_\mu)_1$, *is that* $\mathfrak{Q}$ *be connected.*

The second part of the theorem is a consequence of the first part. We have shown in Theorem I that the condition in the first part of the theorem is sufficient. To show the necessity of the condition, we show that if q_1 and q_2 are not connected, then there is a function μ, uniformly continuous on $\mathfrak{Q}$, such that $\mu(q_1) = 1$, $\mu(q_2) = -1$, and such that 0 does not belong to $(\mathfrak{A}_\mu)_1$.

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* Cf. Hildebrandt, § 19.

In a system $(\mathfrak{Q}; K^1)$, if q_1 and q_2 are not connected there is an $\overline{m}$ such that $q_1 q_2$ are connected for no $m > \overline{m}$. Consider the division of $\mathfrak{Q}$ into subclasses as follows: $\mathfrak{Q}'_1$ is composed of q_1 and all elements q each of which is for some $m > \overline{m}$ connected with q_1. $\mathfrak{Q}'_2$ is composed of q_2 and all elements q each of which is for some $m > \overline{m}$ connected with q_2. $\mathfrak{Q}'_3$ is composed of all elements which are in neither $\mathfrak{Q}'_1$ nor $\mathfrak{Q}'_2$. $\mathfrak{Q}'_1$ and $\mathfrak{Q}'_2$ each contain at least one element. $\mathfrak{Q}'_3$ may contain no elements.

$\mathfrak{Q}'_1$ and $\mathfrak{Q}'_2$ have no common elements. It is clear that neither q_1 nor q_2 can be common to $\mathfrak{Q}'_1$ and $\mathfrak{Q}'_2$. Suppose there is an element q', neither q_1 nor q_2, which is common to $\mathfrak{Q}'_1$ and $\mathfrak{Q}'_2$. q' is connected with q_1 for a certain $m_1 > \overline{m}$. q' is connected with q_2 for a certain $m_2 > \overline{m}$. Call m_0 the smaller of m_1 and m_2. Then for m_0, which exceeds $\overline{m}$, q_1 and q_2 are connected. This contradicts the fact that q_1 and q_2 are connected for no $m > \overline{m}$. From the way $\mathfrak{Q}'_3$ is defined, it can have no elements in common with either $\mathfrak{Q}'_1$ or $\mathfrak{Q}'_2$.

There is no $m > \overline{m}$ for which a q'_1 of $\mathfrak{Q}'_1$ and a q'_2 of $\mathfrak{Q}'_2$ are connected. For suppose there were such an m and such a pair of elements $q'_1 q'_2$. q'_1 is connected with q_1 for $m_1 > \overline{m}$. q'_2 is connected with q_2 for $m_2 > \overline{m}$. Consider m_0 the smaller of m, m_1, m_2. For m_0, q_1 is connected with q'_1, q'_1 with q'_2, and q'_2 with q_2. Therefore q_1 is connected with q_2 for an $m_0 > \overline{m}$. This is a contradiction. Similarly there is no $m > \overline{m}$ for which a q'_1 of $\mathfrak{Q}'_1$ and a q'_3 of $\mathfrak{Q}'_3$ are connected. Neither is there an $m > \overline{m}$ for which a q'_3 of $\mathfrak{Q}'_3$ and a q'_2 of $\mathfrak{Q}'_2$ are connected. Thus, if there is an $m > \overline{m}$ such that the relation $K\hat{q}\tilde{q}m$ holds, then $\hat{q}$ and $\tilde{q}$ must both belong to the same one of the classes $\mathfrak{Q}'_1$, $\mathfrak{Q}'_2$, $\mathfrak{Q}'_3$.

Now consider a function μ such that $\mu(q) = 1$ if q belongs to $\mathfrak{Q}'_1$, and such that $\mu(q) = -1$ if q does not belong to $\mathfrak{Q}'_1$. μ is uniformly continuous on $\mathfrak{Q}$, and the totality of its functional values consists of the two values 1 and —1.

From the previous theorem and the example just given we have the following theorem:

THEOREM V. (a) *A necessary and sufficient condition that every function* μ, *uniformly continuous on an extremal set* $\mathfrak{Q}$ *of a system* $(\mathfrak{Q}; K^1)$, *be such that if* $\mu(q_1) < h < \mu(q_2)$ *then* h *belongs to* $\mathfrak{A}_\mu$, *is that* q_1 *and* q_2 *be connected.* (b) *A necessary and sufficient condition that every function* μ, *uniformly continuous on the extremal set* $\mathfrak{Q}$ *of a system* $(\mathfrak{Q}; K^1)$, *assume every value between each of its values, is that* $\mathfrak{Q}$ *be connected.*

For systems $(\mathfrak{Q}; K)$ which are such that every continuous function on $\mathfrak{Q}$ is likewise uniformly continuous, the preceding theorems hold if in the hypotheses uniform continuity be replaced by continuity. Hildebrandt shows* that in the case of systems $(\mathfrak{Q};\ K^{1367})$, where 3, 6, 7 are properties of the relation K defined in a note below[*] and $\mathfrak{Q}$ is extremal, the continuity of a function μ on $\mathfrak{Q}$ implies its uniform continuity. In view of this, Theorems II, III, V hold for systems $(\mathfrak{Q};\ K^{1367})$ if continuity be substituted for uniform continuity in the hypotheses. In particular we have:

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* Hildebrandt, § 22.

THEOREM VI. *A necessary and sufficient condition that every function* μ *continuous on an extremal set* $\mathfrak{Q}$ *of a system* $(\mathfrak{Q};\ K^{1367})$[†] *assume every value between each pair of its values, is that* $\mathfrak{Q}$ *be connected.*

Hildebrandt also indicates[‡] that a necessary and sufficient condition that every function μ continuous on $\mathfrak{Q}$ of a system $(\mathfrak{Q};\ K^{1367})$ (1) be bounded, (2) attain its bounds, is that $\mathfrak{Q}$ be extremal. Using the results of Hildebrandt and Theorem VI, we have the following theorem:

THEOREM VII. *A necessary and sufficient condition that every function* μ *continuous on* $\mathfrak{Q}$ *of a system* $(\mathfrak{Q};\ K^{1367})$ *shall at the same time be bounded and attain its least upper and greatest lower bounds and every value between these bounds, is that* $\mathfrak{Q}$ *be extremal and connected.*

Among the systems $(\mathfrak{Q};\ K^{1367})$ of Hildebrandt are the systems $(\mathfrak{Q};\ \delta)$ as used by Fréchet.[§]

Various independence considerations might arise here, but we content ourselves by giving two simple examples of systems $(\mathfrak{Q};\ K^{12345678})$[‖] where $\mathfrak{Q}$ is extremal. In the first of these examples $\mathfrak{Q}$ is connected, and in the second $\mathfrak{Q}$ is not connected.

(a) $\mathfrak{Q}$ is the class of all real numbers q such that $0 \leq q \leq 1$. q_1 and q_2 are in the relation Kq_1q_2m if $|q_1 - q_2| \leq \frac{1}{2m}$.

(b) $\mathfrak{Q}$ is the class of all real numbers q such that $0 \leq q \leq 1$ or such that $3 \leq q \leq 4$. q_1 and q_2 are in the relation Kq_1q_2m if $|q_1 - q_2| \leq \frac{1}{2m}$.

DARTMOUTH COLLEGE, *March*, 1913.

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[*] Hildebrandt, § 4.

[†] K has the property 3 if the relation Kq_1q_2m holds for every m only in the case when q_1 and q_2 are identical.

K has the property 5 if there is an integral-valued function, say ϕ_m, which increases indefinitely with m, such that, if we have the relations $Kq_1q_2m\ Kq_2q_3m$, then we have the relation $Kq_1q_3\phi_m$.

K has the property 6 if there is an integral-valued function ϕ_m which increases indefinitely with m, such that, if we have $Kq_2q_1m\ Kq_2q_3m$, then we have $Kq_1q_3\phi_m$. K^5 and K^6 are equivalent if K is symmetrical. Cf. Hildebrandt, § 4.

[‡] Hildebrandt, § 22.

[§] Cf. Hildebrandt, p. 266, foot-note.

[‖] Hildebrandt, § 4.